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A NOTE ON BOUNDING SOLUTIONS FOR CREEPING STRUCTURES SUBJECTED TO LOAD VARIATIONS ABOVE THE SHAKEDOWN LIMIT

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Abstract—Previous work has established upper bounds on the deformations which occur in the cyclic stationary state of a creeping structure. The present note extends the results to describe the transitional behaviour in reaching the cyclic stationary state and also to allow non-cyclic loading. A particular form of the upper bound is applied to experiments reported in the literature of a cylinder subjected to cyclic internal pressure and repeated thermal shocks.

1. INTRODUCTION

The creep deformation of structures subjected to variations of temperature and external loading is considered. For load variations below the short-term shakedown limit, Ponter[1, 2] has used a stress history equal to the elastic stress history plus a constant self-equilibrating stress field to provide upper bounds on the deformation of creeping structures. This method has been extended by Ainsworth[3] to loadings beyond the shakedown limit by using a cyclic elastic–plastic stress history. The resultant bounds[3] describe the behaviour in the cyclic stationary state but do not consider any transient response in reaching this stationary state.

For complex components it is not simple to determine a cyclic elastic–plastic solution particularly for hardening plastic materials and application of the previous work[3] presents difficulties. The present note attempts to overcome this problem by deriving bounds in terms of an elastic–plastic solution which is not necessarily cyclic. The results extend the previous work by describing the transitional response in reaching a cyclic stationary state and also by allowing non-cyclic loading. The results are still limited to simple non-interactive creep and plasticity laws, with yield criteria of linear kinematic hardening and perfect plasticity being considered. A particular form of the upper bound is applied to a pressurised tube to provide a comparison with some experimental results presented by Corum *et al.*[4].

2. MATERIAL BEHAVIOUR

The total strain rate $\dot{\epsilon}_{ij}$ is considered as the sum of four parts

$$\dot{\epsilon}_{ij} = \dot{e}_{ij} + \dot{p}_{ij} + \dot{v}_{ij} + \dot{\theta}_{ij} \tag{1}$$

where $\dot{e}_{ij}$, $\dot{p}_{ij}$, $\dot{v}_{ij}$, $\dot{\theta}_{ij}$ are elastic, plastic, creep and imposed (or thermal) strain rates respectively. Elastic strains are linearly related to stress σ_{ij} giving a positive-definite elastic energy density

$$E(\sigma_{ij}) = \tfrac{1}{2}\sigma_{ij}e_{ij}. \tag{2}$$

A linear kinematic strain hardening model of plasticity is used with a yield criterion

$$f(\sigma_{ij} - \alpha_{ij}) \leq \sigma_y \tag{3}$$

where σ_y is the virgin yield stress. The shift of the yield surface is related to plastic strain rate by

$$\dot{\alpha}_{ij} = c\dot{p}_{ij} \tag{4}$$

where c is a positive constant assumed independent of temperature and time. Plastic strain rates are taken normal to the yield surface which is assumed convex so that

$$\{(\sigma_{ij} - \alpha_{ij}) - (\sigma_{ij}^* - \alpha_{ij}^*)\}\dot{p}_{ij} \geq 0 \tag{5}$$

where $(\sigma_{ij}^* - \alpha_{ij}^*)$ is any state satisfying inequality (3).

Creep strain rates are taken normal to the dissipation function, $\dot{D}$,

$$\dot{D}(\sigma_{ij}) = \sigma_{ij}\dot{v}_{ij} = \sigma_0 \dot{v}_0 \phi^{n+1}(\sigma_{ij}/\sigma_0)g(\theta) \tag{6}$$

where n, σ_0, $\dot{v}_0$ are constants and $g(\theta)$ is a positive function of temperature θ. The function ϕ is convex and homogeneous of degree one in (σ_{ij}/σ_0) reducing to the value unity for an uniaxial stress σ_0. Since ϕ is convex and homogeneous of degree one in stress,

$$n(\sigma_{ij}^* - \sigma_{ij})\dot{v}_{ij} \leq \dot{D}\{n\sigma_{ij}^*/(n+1)\} \tag{7}$$

where $\dot{v}_{ij}$ is the creep strain rate at stress σ_{ij} and temperature θ and $\dot{D}(\sigma_{ij}^*)$ is the dissipation rate corresponding to any stress σ_{ij}^* at the same temperature θ (see, e.g. [1]).

3. UPPER BOUNDS

Consider a body of volume V and surface S with negligible body forces. The temperature θ of the body and the imposed strains θ_{ij} are given functions of time, t, and position. The body is subjected to a given history of loading $P_i(t)$ over part of the surface and to zero surface velocities over the remainder of S. All deformations are assumed small so that changes in geometry can be neglected. The material of the body has the elastic, plastic and creep properties described in Section 2 and the stresses, strains and displacements resulting from the loading are denoted by unstarred quantities.

The behaviour of the structure is bounded below by the behaviour of the same structure composed of material with the same elastic–plastic properties but which does not creep. For the elastic–plastic analysis the same temperature θ and imposed strains θ_{ij} are used but the mechanical loading is increased to $P_i(t) + R_i(t)$ where the additional loads R_i can be chosen arbitrarily. Quantities resulting from the elastic–plastic analysis without creep are denoted by starred quantities.

From the principle of virtual work

$$\int_0^T \int_S R_i(\dot{u}_i - \dot{u}_i^*)\,dS\,dt = \int_0^T \int_V (\sigma_{ij}^* - \sigma_{ij})(\dot{\varepsilon}_{ij} - \dot{\varepsilon}_{ij}^*)\,dV\,dt$$

where $\dot{u}_i$ are displacement rates and T is some time of interest. Splitting the total strain rates into components using eqn (1) this becomes

$$\int_0^T \int_S R_i(\dot{u}_i - \dot{u}_i^*)\,dS\,dt = \int_0^T \int_V (\sigma_{ij}^* - \sigma_{ij})(\dot{e}_{ij} - \dot{e}_{ij}^* + \dot{p}_{ij} - \dot{p}_{ij}^* + \dot{v}_{ij})\,dV\,dt \tag{8}$$

since the imposed strains are identical. From inequality (5) and eqn (4)

$$(\sigma_{ij}^* - \sigma_{ij})(\dot{p}_{ij} - \dot{p}_{ij}^*) \leq -\frac{1}{2c}\frac{d}{dt}\{(\alpha_{ij}^* - \alpha_{ij})(\alpha_{ij}^* - \alpha_{ij})\}. \tag{9}$$

Equation (8) then becomes on noting (2), (7) and (9)

$$\int_0^T \int_S R_i\dot{u}_i\,dS\,dt \leq \int_0^T \int_S R_i\dot{u}_i^*\,dS\,dt + A(0) - A(T) + \frac{1}{n}\int_0^T \int_V \dot{D}\left(\frac{n\sigma_{ij}^*}{n+1}\right)dV\,dt \tag{10}$$

where

$$A = \int_V \{E(\sigma_{ij}^* - \sigma_{ij}) + \tfrac{1}{2}(\alpha_{ij}^* - \alpha_{ij})(\alpha_{ij}^* - \alpha_{ij})/c\}\, dV. \tag{11}$$

In general, $A(T)$ will be unknown but since A is positive the bound (10) is still valid with $A(T)$ omitted. The actual displacements u_i are then bounded by the initial conditions and by the elastic–plastic solution. Consequently, a detailed analysis of the time-dependent creep deformation is not required. By choice of the additional loads R_i inequality (10) can provide bounds on particular displacements, deformations or work. As in [3] the bound may be optimised by choice of the magnitude of the additional loading and by choice of the initial conditions $\sigma_{ij}^*(0)$, $\alpha_{ij}^*(0)$.

For cyclic loading, a steady cyclic state is reached (Frederick and Armstrong[5], Ainsworth[3]) and the quantity A of eqn (11) becomes periodic. If times 0, T are identified as the beginning and end of a cycle in the steady cyclic state then $A(0) = A(T)$, σ_{ij}^* is a cyclic plasticity solution and inequality (10) reduces to the bound of [3].

For application to the problem considered in Section 4, the bound (10) is specialised by taking R_i as a constant load R so providing a bound on the displacement u in the line of R. The actual initial state is $\sigma_{ij}(0) = \alpha_{ij}(0) = u(0) = 0$. Optimisation for the initial state $\sigma_{ij}^*(0)$, $\alpha_{ij}^*(0)$ is not attempted. Instead $\alpha_{ij}^*(0)$ is taken as zero so that $\sigma_{ij}^*(0)$ is simply the stress field resulting elastically from the load R (which is insufficient to cause yielding). The term $A(T)$ is unknown but can be omitted from the bound as it is positive. The term $A(0)$ is

$$A(0) = \int_V \frac{1}{2}\sigma_{ij}^*(0)e_{ij}^*(0)\, dV = \frac{1}{2}Ru^*(0)$$

by the principle of virtual work. The bound (10) then becomes

$$u(T) \le u^*(T) - \frac{1}{2}u^*(0) + (1/nR)\int_0^T \int_V \dot{D}\{n\sigma_{ij}^*/(n+1)\}\, dV\, dt \tag{12}$$

which can be optimised for the additional load R.

4. COMPARISON OF UPPER BOUND AND EXPERIMENT

Corum *et al.*[4], have reported tests on straight sections of type 304 stainless steel pipe. The pipes were subjected to thermal shocks followed by dwell periods under internal pressure at 1100°F as shown schematically in Fig. 1. The test considered here is denoted TTTF-1 by Corum *et al.*[4], and consisted of thirteen nominally identical cycles with a peak pressure of 700 psi and a hold time of 160 hr. The temperature profiles resulting from the thermal shocks are presented in Fig. 7 of [4] and may be represented by

$$\theta(r) = \theta_0 - \Delta\theta(r-b)^2/(b-a)^2 \tag{13}$$

where θ_0 is a constant; a, b are internal and external radii, respectively; and θ is the

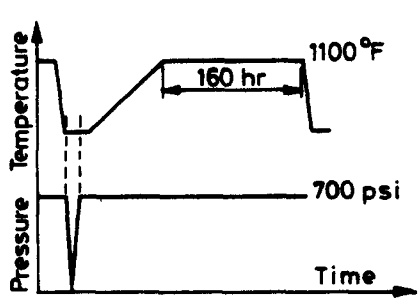

Fig. 1. Nominal cycle.

temperature at radius r. The peak value of the temperature difference measured experimentally was $\Delta\theta = 175°F$.

The pipe was thin-walled with a ratio of mean radius to thickness of 10.8. Consequently, the temperature field (13) produces stress fields $\sigma_x \simeq \sigma_\phi$ and $\sigma_r \ll \sigma_\phi$ where σ_x, σ_r, σ_ϕ are axial, radial and hoop stress components, respectively. For the purposes of analysis, the pipe is represented by an uniaxial model used by Bree[6] and others. The equi-biaxial behaviour under thermal loading is modelled by using an elastic modulus $E/(1-\nu)$ and a plastic modulus $2E_p$ where ν is Poisson's ratio and E, E_p are the elastic and plastic moduli in an uniaxial test. In the uniaxial model the through-thickness strain is assumed constant and represents the mean hoop strain in the pipe. An applied stress represents the hoop stress arising from the pressure. Details of the elastic–plastic analysis are omitted as it closely follows the analysis of Bree[6] except that the parabolic temperature field (13) replaces the linear variation used by Bree and the pressure also varies. The numerical methods used in the elastic–plastic solution are similar to those described by Ainsworth[7].

To calculate the upper bound a constant additional pressure R has been applied so that inequality (12) bounds the hoop strain by the calculated strain u^* in the uniaxial model and by creep strains resulting from the calculated stress history σ^*. The elastic–plastic solution has been obtained using material properties taken from Corum[8] and these are listed in Table 1. Where necessary, mean values over the temperature range have been used. Kinematic hardening behaviour has been assumed with the tenth cycle data of [8] used for all cycles except the first when first cycle data was used. The creep contribution to the bound (12) only occurs during the dwell periods at high temperature. During these dwell periods the elastic–plastic stress σ^* is constant at any point so that $\int \dot{D}\,dt$ over a cycle is simply the product of $n\sigma^*/(n+1)$ and the creep strain corresponding to $n\sigma^*/(n+1)$ in the dwell period. Creep strains have been obtained directly from the data presented by Corum[8]. A stress index $n = 2.8$ fits this data for the times and stress range of interest. The value of σ^* during the dwell period changes from cycle to cycle and the total time integral required in (12) has been obtained from the sum of the cycles using a time-hardening rule.

The value of additional pressure R which minimised the bound (12) for times greater than 160 hr (one cycle) was found to be 150 psi which is about 20% of the applied pressure of 700 psi. However, the bound is not particularly sensitive to R and varies by less than 20% for R between 10% and 50% of the applied pressure. A single value of R has been used although a slightly better bound during the first cycle might have been obtained using a smaller value of R. For loading below the shakedown limit, Ponter[9] has suggested that the additional loading to minimise the upper bound is approximately given by $R = \{\max P(t)\}/n$. In the present case this gives a value $R = 36\%$ of the pressure which is greater than the optimum value but which would still provide a reasonable bound.

The upper bound (12) is compared with the experimental results of Corum *et al.*[4] in Fig. 2. It can be seen that the upper bound is in very good agreement with experiment. After 13 cycles the contributions to the bound (12) are 0.17% from the creep term and 0.21% from the non-creep terms. Thus the non-creep terms are a substantial part of the total bound during the transient stage prior to reaching a cyclic stationary state. However, for longer term applications the non-creep terms

Table 1. Material properties used in the analysis of pressurised tube

Quantity		Value
Poisson's ratio	ν	0.3
Elastic modulus	E	23.3 10^6 psi*
Plastic modulus	E_p	0.67 10^6 psi
Product of E and the coefficient of thermal expansion	$E\alpha$	256.7 psi/°F
First cycle yield stress		10.4 10^3 psi*
Tenth cycle yield stress		13.0 10^3 psi

*Denotes an average value over the temperature range.

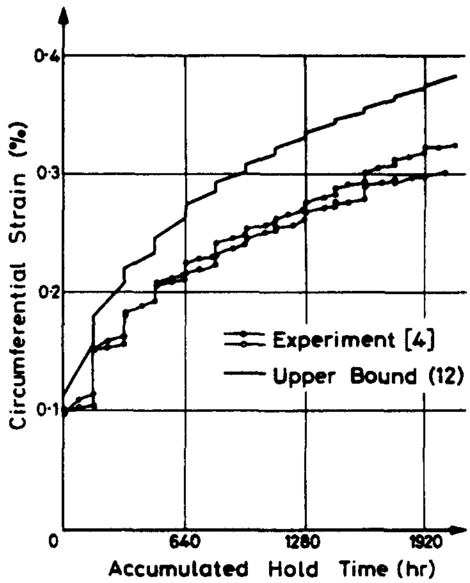

Fig. 2. Comparison of experiment and upper bound.

become less significant and reach a maximum of 0.25% whereas the creep contribution continues to increase with time.

It should be noted that the upper bound has no allowance for the effect on material properties of any creep-plastic interactions. However, a full analysis of the problem[10] using the same material data[18] has shown almost exact agreement with experiment. This suggests that any creep-plastic interactions have a negligible effect on the overall deformation in the present case and that use of the independently obtained plastic and creep data is justified. In general, however, the possibility of creep-plastic interactions should be considered when using the upper bound predictions as, indeed, they should be considered in a full analysis.

5. CONCLUDING REMARKS

Upper bounds on the deformation of creeping structures operating above the short-term shakedown limit have been extended to describe the transitional response in reaching a cyclic stationary state and to allow the use of non-cyclic elastic–plastic solutions. Although the bounding method requires an inelastic analysis to determine the elastic–plastic response, detailed analysis of the time-dependent creep deformation is eliminated. The upper bound has been shown to be in good agreement with experimental results reported in the literature for a pressurised cylinder subjected to repeated thermal shocks.

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